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On a maximum principle for minimal surfaces and their integrable discrete counterparts

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Abstract

A novel maximum principle for both classical and discrete minimal surfaces is recorded. In the discrete setting, the maximum principle is based on purely geometric notions of discrete Gaußian and mean curvatures and parallel discrete surfaces. As an additional confirmation of the validity of these notions, a discrete analogue of a classical theorem for linear Weingarten surfaces is obtained. Connections with the 'parallel surface method' utilized in condensed matter physics are discussed.

Keywords: Discrete differential geometry; Integrable systems; Minimal surfaces

1. Introduction

In the past decade, the study of canonical discretizations of differential geometries which preserve the integrability of their underlying nonlinear differential equations has been a subject of extensive research. To a large extent, this area of *discrete differential geometry* has been initiated by the pioneering results on integrable discretizations of pseudospherical and isothermic surfaces by Bobenko and Pinkall [2,3]. However, there exist strong connections with earlier work on both 'difference geometry' by Sauer [15,16] and Wunderlich [18] and the classical differential geometry of Bäcklund transformations and associated permutabil-

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ity theorems by Bianchi, Darboux, Demoulin, Ribaucour and others (see the monograph *Discrete Integrable Geometry and Physics* [6] for introductory references).

In the context of a large class of integrable discrete surfaces ('O surfaces'), definitions of Gaußian and mean curvatures for canonical discrete analogues of surfaces parametrized in terms of curvature coordinates have been proposed in [17]. In particular, it has been shown that discrete surfaces of constant Gaußian or mean curvature are governed by integrable difference equations and admit associated Bäcklund transformations [1,14]. Moreover, it has been demonstrated that the class of discrete surfaces of vanishing mean curvature (discrete minimal surfaces) coincides with the class of discrete isothermic minimal surfaces discussed in [3]. The latter class has been defined in terms of a discrete Christoffel transform and admits a discrete analogue of the classical Weierstrass representation of minimal surfaces [8].

In the present paper, we first set down a novel maximum principle which characterizes classical minimal surfaces. This is achieved by regarding any given surface as being embedded in a layer of parallel surfaces. As a by-product, we obtain a characterization of the mean curvature of a surface which may immediately be generalized to the discrete case if an appropriate notion of parallel discrete surfaces is introduced. It turns out that this geometric definition of discrete mean curvature coincides with that justified algebraically in [17]. Moreover, we set down a discrete analogue of a classical theorem of differential geometry involving 'linear Weingarten surfaces' [8]. We then demonstrate that, remarkably, the maximum principle for minimal surfaces holds *mutatis mutandis* in the discrete setting. This highlights once again the canonicity of both the standard integrable discretization of minimal surfaces and the definitions of the discrete Gaußian and mean curvatures. In the last section, we establish connections with the 'parallel surface method' which has been developed in the context of the determination of interfacial curvatures in condensed matter systems [13].

2. A maximum principle for minimal surfaces

In the following, we are concerned with the differential geometry of surfaces in a three-dimensional Euclidean ambient space. In particular, we focus on classical minimal surfaces. Even though minimal surfaces are known to admit a variety of characteristic properties [7] such as vanishing mean curvature or vanishing first variation of the area functional, we here propose another characterization which turns out to be canonical in the discrete setting to be considered in Section 3. In this connection, it proves convenient to parametrize the surfaces in terms of curvature coordinates (x, y). Thus, if $\boldsymbol{r}$ denotes the position vector of a surface Σ and $\boldsymbol{N} = (\boldsymbol{r}_x \times \boldsymbol{r}_y)/|\boldsymbol{r}_x \times \boldsymbol{r}_y|$ is the corresponding unit normal then the first and second fundamental forms $\mathrm{I} = \mathrm{d}\boldsymbol{r}^2$ and $\mathrm{II} = -\mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{N}$ become [8]:

$$\mathrm{I} = H^2\,\mathrm{d}x^2 + K^2\,\mathrm{d}y^2, \qquad \mathrm{II} = \kappa_1 H^2\,\mathrm{d}x^2 + \kappa_2 K^2\,\mathrm{d}y^2. \tag{2.1}$$

Here, κ_1 and κ_2 designate the usual principal curvatures.

In order to establish a maximum principle for minimal surfaces, we regard a surface Σ as being embedded in a layer of parallel surfaces $\Sigma^{\parallel}(c)$, where the parameter c denotes the distance to the surface $\Sigma^{\|}(0) = \Sigma$, so that

$$\boldsymbol{r}^{\|}(c) = \boldsymbol{r} + c\boldsymbol{N}. \tag{2.2}$$

By virtue of the formulae of Rodrigues [8]:

$$\boldsymbol{N}_x = -\kappa_1 \boldsymbol{r}_x, \qquad \boldsymbol{N}_y = -\kappa_2 \boldsymbol{r}_y, \tag{2.3}$$

it is readily shown that the corresponding one-parameter family of metrics $\mathrm{I}^{\|}(c)$ adopts the form

$$\mathrm{I}^{\|} = H^{\|2}\,\mathrm{d}x^2 + K^{\|2}\,\mathrm{d}y^2 \tag{2.4}$$

with the metric coefficients $H^{\|}$ and $K^{\|}$ given by

$$H^{\|} = (1 - c\kappa_1)H, \qquad K^{\|} = (1 - c\kappa_2)K. \tag{2.5}$$

Moreover, the second fundamental forms turn out to be likewise purely diagonal and hence the coordinates x and y parametrize the lines of curvature on all members of the one-parameter family of surfaces.

Since, for any surface Σ, the associated one-parameter family of parallel surfaces $\Sigma^{\|}$ shares the normal congruence, the infinitesimal surface elements $d\Sigma^{\|}$ of area $\mathrm{d}A^{\|}$ corresponding to fixed parameters x and y generate an infinitesimal 'tube' bounded by normal lines. It is therefore natural to define the *dilation factor* $Q(c)$ as (cf. [9])

$$Q = \frac{\mathrm{d}A^{\|}}{\mathrm{d}A} = \frac{\sqrt{\det \mathrm{I}^{\|}}}{\sqrt{\det \mathrm{I}}}, \tag{2.6}$$

which provides a measure of the variation of the 'local area' along a normal line relative to the surface Σ. Evaluation of the dilation factor yields

$$Q = \frac{H^{\|}K^{\|}}{HK} = 1 - 2c\mathcal{M} + c^2\mathcal{K}, \tag{2.7}$$

where

$$\mathcal{M} = \frac{\kappa_1 + \kappa_2}{2}, \quad \mathcal{K} = \kappa_1\kappa_2 \tag{2.8}$$

are the mean and Gaußian curvatures respectively associated with the surface Σ. It is apparent that $c = 0$ is a stationary point of the dilation factor if and only if $\mathcal{M} = 0$, that is if and only if Σ constitutes a minimal surface. Moreover, since $\mathcal{K} < 0$ for (non-developable) minimal surfaces, the following maximum principle is evident.

Theorem 1. *A surface Σ is a minimal surface if and only if the corresponding dilation factor $Q(c)$ is stationary on Σ with respect to c. In this case, the dilation factor attains a local maximum on Σ.*

In conclusion, it is noted that the relation (2.7) delivers a simple geometric expression for the mean curvature of a surface, namely

$$\mathcal{M} = \frac{\mathrm{d}A^{\|}(-c) - \mathrm{d}A^{\|}(c)}{4c\,\mathrm{d}A}. \tag{2.9}$$

In fact, as shown in the next section, the latter may be used as a basis for the definition of *discrete* mean curvature.

3. Parallel discrete surfaces and discrete mean curvature

In order to establish a maximum principle for discrete minimal surfaces, it is required to introduce a notion of parallelism for discrete surfaces. Here, a discrete surface Σ is defined as a two-dimensional lattice of $\mathbb{Z}^2$ combinatorics in a three-dimensional Euclidean space or, on identification of a discrete surface with its position vector, a map

$$\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3, \quad (n_1, n_2) \mapsto \boldsymbol{r}(n_1, n_2). \tag{3.1}$$

Moreover, we assume that the quadrilaterals $[\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_{12}, \boldsymbol{r}_2]$ of any discrete surface may be inscribed in circles. Here, subscripts indicate unit increments of the discrete variables so that

$$\begin{aligned} &\boldsymbol{r} = \boldsymbol{r}(n_1, n_2), \quad \boldsymbol{r}_1 = \boldsymbol{r}(n_1 + 1, n_2), \quad \boldsymbol{r}_2 = \boldsymbol{r}(n_1, n_2 + 1), \\ &\boldsymbol{r}_{12} = \boldsymbol{r}(n_1 + 1, n_2 + 1). \end{aligned} \tag{3.2}$$

Discrete surfaces which obey the above 'cyclicity' condition have been recognized as canonical discrete analogues of surfaces parametrized in terms of curvature coordinates (see [6] and references therein) and are frequently referred to as 'discrete curvature nets.' However, since we are concerned exclusively with discrete curvature nets, we simply refer to them as discrete surfaces.

The cyclicity property now guarantees that the following definition is meaningful (cf. Fig. 1).

Definition 1. A discrete surface $\boldsymbol{r}^{\parallel} : \mathbb{Z}^2 \to \mathbb{R}^3$ is considered *parallel* and at a *distance* c to a discrete surface $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3$ if the 'vertical' quadrilaterals $[\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_1^{\parallel}, \boldsymbol{r}^{\parallel}]$ and $[\boldsymbol{r}, \boldsymbol{r}_2, \boldsymbol{r}_2^{\parallel}, \boldsymbol{r}^{\parallel}]$ constitute isosceles trapezoids of edge length c.

In the continuous case, any surface (locally) admits two parallel surfaces which are at a distance c. In the discrete case, for any given c, there exists a two-parameter family of parallel discrete surfaces. Indeed, if a single vertex $\boldsymbol{r}^{\parallel}$ is arbitrarily prescribed subject to $|\boldsymbol{r}^{\parallel} - \boldsymbol{r}| = c$ then the parallel discrete surface is uniquely determined by the condition of vertical isosceles trapezoidal quadrilaterals. Even though this fact may be deduced directly

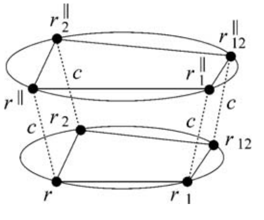

Fig. 1. The definition of parallel discrete surfaces.

by geometric means, it is enlightening to make use of the discrete Gauß map introduced in [17]. Thus, given a discrete surface Σ with position vector $\boldsymbol{r}$ and a point $\boldsymbol{N}(0,0)$ on the unit sphere S^2, we may construct another two points $\boldsymbol{N}(1,0)$ and $\boldsymbol{N}(0,1)$ on S^2 by demanding that the line segments $[\boldsymbol{N}(0,0), \boldsymbol{N}(1,0)]$ and $[\boldsymbol{N}(0,0), \boldsymbol{N}(0,1)]$ be parallel to the edges $[\boldsymbol{r}(0,0), \boldsymbol{r}(1,0)]$ and $[\boldsymbol{r}(0,0), \boldsymbol{r}(0,1)]$ respectively. A fourth point $\boldsymbol{N}(1,1)$ is defined as the point of intersection of the two lines which are parallel to the edges $[\boldsymbol{r}(1,0), \boldsymbol{r}(1,1)]$ and $[\boldsymbol{r}(0,1), \boldsymbol{r}(1,1)]$ and pass through the points $\boldsymbol{N}(1,0)$ and $\boldsymbol{N}(0,1)$ respectively. The cyclicity of the discrete surface Σ then guarantees that the fourth point also lies on S^2 as illustrated in Fig. 2. Accordingly, iterative application of this procedure generates another discrete surface $\Sigma_\circ$ on the unit sphere which is uniquely determined by the choice of $\boldsymbol{N}(0,0)$. The discrete surface $\Sigma_\circ$ may be regarded as a ‘spherical representation’ of the discrete surface Σ or a discrete analogue of the Gauß map [8] with $\boldsymbol{N}$ being a discrete ‘normal.’ Furthermore, it is evident that any parallel discrete surface admits the representation

$$\boldsymbol{r}^{\parallel} = \boldsymbol{r} + c\boldsymbol{N}. \tag{3.3}$$

In this connection, it is convenient to regard c as a distance parameter which may also be negative. It is also observed that for any discrete surface and an associated spherical representation, any infinite sequence of distance parameters

$$\cdots < c_{-2} < c_{-1} < c_0 = 0 < c_1 < c_2 < \cdots \tag{3.4}$$

generates a lattice of $\mathbb{Z}^3$ combinatorics which has the property that all ‘horizontal’ and ‘vertical’ quadrilaterals may be inscribed in circles. The latter has been widely accepted as a defining property for discrete orthogonal coordinate systems [4,6]. It is noted that, in the continuous setting, the triple (x, y, c) indeed defines an orthogonal coordinate system in $\mathbb{R}^3$.

The existence of discrete Gauß maps gives rise to the following natural definition (cf. Fig. 2).

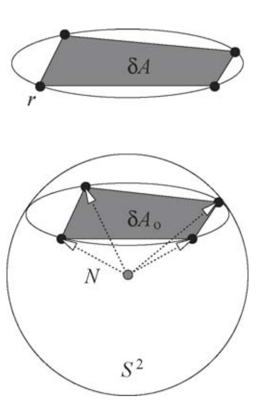

Fig. 2. A spherical representation of a discrete surface.

Definition 2. The discrete *Gaußian curvature* of a discrete surface Σ with respect to an associated spherical representation $\Sigma_\circ$ is defined by

$$\mathcal{K} = \frac{\delta A_\circ}{\delta A}, \tag{3.5}$$

where δA and $\delta A_\circ$ denote the oriented areas of corresponding quadrilaterals of Σ and $\Sigma_\circ$ respectively.

As in the classical continuous case, the (local) discrete Gaußian curvature is taken to be negative if the parallel quadrilaterals $[\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_{12}, \boldsymbol{r}_2]$ and $[\boldsymbol{N}, \boldsymbol{N}_1, \boldsymbol{N}_{12}, \boldsymbol{N}_2]$ are of opposite orientation. This situation is illustrated in Fig. 3. In [17], it has been shown that discrete surfaces of constant Gaußian curvature are integrable in the sense that these admit a Bäcklund transformation based on a Lax pair for the underlying difference equations. This fact is well known in the classical continuous setting (see [14] and references therein).

In analogy with (2.9), the mean curvature of a discrete surface may now be given in terms of the two parallel discrete surfaces which are at a distance $\pm c$ with respect to a spherical representation (cf. Fig. 4).

Definition 3. The discrete *mean curvature* of a discrete surface Σ with respect to an associated spherical representation $\Sigma_\circ$ is defined by

$$\mathcal{M} = \frac{\delta A^{\|}(-c) - \delta A^{\|}(c)}{4c\delta A}, \tag{3.6}$$

where δA and $\delta A^{\|}(\pm c)$ denote the oriented areas of corresponding quadrilaterals of Σ and $\Sigma^{\|}(\pm c)$ respectively with $\boldsymbol{r}^{\|}(c) = \boldsymbol{r} + c\boldsymbol{N}$.

In Section 4, it is demonstrated that the above geometric definition is independent of the parameter c and coincides with the algebraic definition of the mean curvature proposed in [17]. Accordingly, constant discrete mean curvature surfaces and discrete minimal surfaces ($\mathcal{M} = 0$) are likewise integrable. The geometric meaning of the condition $\mathcal{M} = 0$ is evident. In this connection, it is observed that it is not difficult to show that the spherical representation associated with any (generic) constant discrete mean curvature surface or discrete minimal surface is unique. In fact, under the admissible and geometrically natural assumption that the cross-ratio of the quadrilaterals is constant (cf. [3]), this particular spherical representation

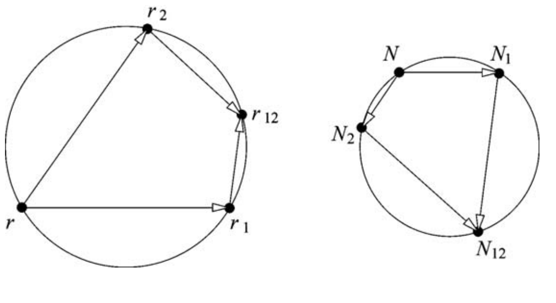

Fig. 3. A cyclic quadrilateral and a spherical representation of opposite orientation.

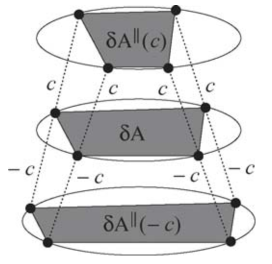

Fig. 4. The parallel discrete surfaces Σ, $\Sigma^{\parallel}(c)$ and $\Sigma^{\parallel}(-c)$ employed in the definition of the discrete mean curvature $\mathcal{M}$.

may be characterized in the following manner. In the case of discrete minimal surfaces, the spherical representation is given by

$$\boldsymbol{N} = \frac{(\boldsymbol{r}_1 - \boldsymbol{r}_{\bar{1}}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_{\bar{2}})}{|(\boldsymbol{r}_1 - \boldsymbol{r}_{\bar{1}}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_{\bar{2}})|}, \tag{3.7}$$

where $\boldsymbol{r}_{\bar{1}} = \boldsymbol{r}(n_1 - 1, n_2)$ and $\boldsymbol{r}_{\bar{2}} = \boldsymbol{r}(n_1, n_2 - 1)$. This is in agreement with the discrete Gauß map proposed in [3] on which the analogue of the classical Weierstrass representation for discrete minimal surfaces is based. In the case of non-vanishing discrete mean curvature, one may define, as in the continuous setting, the 'centres of mean curvature spheres' $\boldsymbol{S}$ by

$$\boldsymbol{S} = \boldsymbol{r} + \frac{1}{\mathcal{M}}\boldsymbol{N} \tag{3.8}$$

and show that these coincide with those algebraically obtained in [5].

4. A maximum principle for discrete minimal surfaces

We now demonstrate that, remarkably, the maximum principle stated in Theorem 1 is, *mutatis mutandis*, also valid in the discrete setting. Here, in view of the formal continuum limit, we assume that the quadrilaterals of a discrete surface are *non-degenerate* and *embedded*, that is the edges of any quadrilateral do not intersect. Once again, we regard a discrete surface Σ as being a member of a one-parameter family of parallel surfaces $\Sigma^{\parallel}(c)$ given by

$$\boldsymbol{r}^{\parallel} = \boldsymbol{r} + c\boldsymbol{N}, \tag{4.1}$$

where $\boldsymbol{N}$ constitutes any (but fixed) discrete Gauß map. Furthermore, the canonical discrete analogue of the dilation factor $Q(c)$ is defined by

$$Q = \frac{\delta A^{\parallel}}{\delta A}. \tag{4.2}$$

In order to prove the maximum principle, we first introduce a suitable algebraic framework. In particular, the algebraic expressions for the discrete Gaußian and mean curvatures presented in [17] are retrieved. As a by-product, a discrete analogue of a classical theorem for linear Weingarten surfaces is obtained.

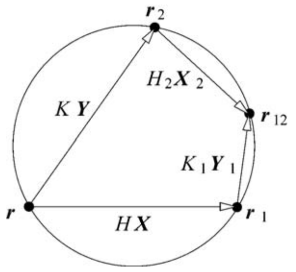

Fig. 5. The ‘frame’ attached to an embedded quadrilateral.

4.1. Algebraic preliminaries: discrete linear Weingarten surfaces

The edge vectors of a discrete surface Σ with position vector $\boldsymbol{r}$ are naturally decomposed into

$$\boldsymbol{r}_1 - \boldsymbol{r} = H\boldsymbol{X}, \qquad \boldsymbol{r}_2 - \boldsymbol{r} = K\boldsymbol{Y}, \tag{4.3}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are unit ‘tangent’ vectors (cf. Fig. 5). Since the quadrilaterals are planar, the tangent vectors $\boldsymbol{X}_2$ and $\boldsymbol{Y}_1$ may be expressed as linear combinations of the tangent vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$. In the case of cyclic quadrilaterals, one obtains the relations [17]:

$$\boldsymbol{X}_2 = \frac{\boldsymbol{X} + q\boldsymbol{Y}}{\Gamma}, \qquad \boldsymbol{Y}_1 = \frac{\boldsymbol{Y} + p\boldsymbol{X}}{\Gamma}, \quad \Gamma = \sqrt{1 - pq}, \tag{4.4}$$

where the quantities p and q are related to the oriented edge lengths H, K and H_2, K_1 by

$$H_2 = \frac{H + pK}{\Gamma}, \qquad K_1 = \frac{K + qH}{\Gamma}. \tag{4.5}$$

Here, the orientation of the tangent vectors has been chosen in such a way that

$$\boldsymbol{X}_2 \times \boldsymbol{Y} + \boldsymbol{Y}_1 \times \boldsymbol{X} = \boldsymbol{0}. \tag{4.6}$$

Moreover, the cyclicity of the quadrilaterals is encoded in the relation

$$\boldsymbol{X}_2 \cdot \boldsymbol{Y} + \boldsymbol{Y}_1 \cdot \boldsymbol{X} = 0, \tag{4.7}$$

which confirms that opposite angles in an embedded cyclic quadrilateral add up to π. These two relations imply that a quadrilateral is embedded and non-degenerate if and only if

$$HH_2KK_1 > 0. \tag{4.8}$$

The latter property gives rise to the expression

$$|\delta A| = |H_2K + K_1H|\frac{|\boldsymbol{X} \times \boldsymbol{Y}|}{2\Gamma} \tag{4.9}$$

for the area of an embedded cyclic quadrilateral.

By construction, the edges of a spherical representation $\Sigma_\circ$ are parallel to the corresponding edges of a discrete surface Σ. Accordingly, the decomposition

$$\boldsymbol{N}_1 - \boldsymbol{N} = H_\circ\boldsymbol{X}, \qquad \boldsymbol{N}_2 - \boldsymbol{N} = K_\circ\boldsymbol{Y} \tag{4.10}$$

obtains, where the lattice functions $H_\circ$ and $K_\circ$ likewise obey the system (4.5), namely

$$H_{\circ 2} = \frac{H_\circ + pK_\circ}{\Gamma}, \qquad K_{\circ 1} = \frac{K_\circ + qH_\circ}{\Gamma}. \tag{4.11}$$

The relation (4.9) combined with its analogue

$$|\delta A_\circ| = |H_{\circ 2}K_\circ + K_{\circ 1}H_\circ| \frac{|\boldsymbol{X} \times \boldsymbol{Y}|}{2\Gamma} \tag{4.12}$$

therefore delivers the algebraic expression

$$\mathcal{K} = \frac{H_{\circ 2}K_\circ + K_{\circ 1}H_\circ}{H_2 K + K_1 H} \tag{4.13}$$

for the discrete Gaußian curvature (3.5), wherein the sign reflects the correct relative orientation of the quadrilaterals. It is observed in passing that this form of the discrete Gaußian curvature may also be employed in the case of non-embedded quadrilaterals.

By virtue of (4.1), the 'metric' coefficients $H^\parallel$ and $K^\parallel$ of a parallel discrete surface $\Sigma^\parallel$ are related to those of the discrete surface Σ and a corresponding spherical representation $\Sigma_\circ$ by

$$H^\parallel = H + cH_\circ, \qquad K^\parallel = K + cK_\circ. \tag{4.14}$$

On taking into account the relative orientation of the quadrilaterals, evaluation of the discrete mean curvature (3.6) produces

$$\mathcal{M} = -\frac{H_2 K_\circ + K_1 H_\circ + H_{\circ 2} K + K_{\circ 1} H}{2(H_2 K + K_1 H)}, \tag{4.15}$$

which is indeed independent of the distance c and coincides with the algebraic definition of the discrete mean curvature proposed in [17]. It is noted that less symmetric forms of the discrete mean curvature are given by

$$\mathcal{M} = -\frac{H_2 K_\circ + K_1 H_\circ}{H_2 K + K_1 H} = -\frac{H_{\circ 2} K + K_{\circ 1} H}{H_2 K + K_1 H}. \tag{4.16}$$

The expressions (4.13) and (4.15) may readily be shown to deliver the relations

$$\mathcal{K}^\parallel = \frac{\mathcal{K}}{1 - 2c\mathcal{M} + c^2\mathcal{K}}, \qquad \mathcal{M}^\parallel = \frac{\mathcal{M} - c\mathcal{K}}{1 - 2c\mathcal{M} + c^2\mathcal{K}} \tag{4.17}$$

between the Gaußian and mean curvatures of a discrete surface and an associated parallel discrete surface at distance c. Remarkably, these coincide with those known in classical differential geometry [8]. Moreover, if the discrete surface Σ is of constant Gaußian curvature

$$\mathcal{K} = \pm\frac{1}{\rho^2}, \quad \rho = \text{constant} \tag{4.18}$$

then there exists a *linear* relation between the discrete Gaußian and mean curvatures of the parallel discrete surface $\Sigma^\parallel$, namely

$$(c^2 \mp \rho^2)\mathcal{K}^\parallel + 2c\mathcal{M}^\parallel + 1 = 0. \tag{4.19}$$

The latter encodes a discrete analogue of a well-known theorem of classical differential geometry [8].

Theorem 2. *Any discrete surface which is parallel to a discrete surface of constant Gaußian curvature constitutes a discrete linear Weingarten surface.*

By definition, Weingarten surfaces are surfaces for which the Gaußian and mean curvatures are functionally dependent. In the current situation, the functional dependence is linear. Even though discrete linear Weingarten surfaces have been shown in [17] to be amenable to the techniques of soliton theory, the above theorem implies that, as in the continuous case, their integrability is inherited from discrete surfaces of constant Gaußian curvature. It is noted that, in the case of positive constant Gaußian curvature, the above result encapsulates a discrete analogue of a classical theorem due to Bonnet [8] in that the parallel discrete surfaces $\Sigma(\pm\rho)$ are of constant mean curvature $\mp 1/2\rho$ [3,17].

4.2. A maximum principle

Evaluation of (4.9) for a discrete surface and an associated parallel surface is readily shown to lead to the following expression for the dilation factor Q.

Theorem 3. *The dilation factor associated with a discrete surface Σ and a spherical representation $\Sigma_\circ$ is given by*

$$Q = 1 - 2c\mathcal{M} + c^2\mathcal{K}, \tag{4.20}$$

which is identical in form to the expression (2.7) valid in the continuous setting.

The maximum principle now reads as follows.

Corollary 1. *A discrete surface Σ is a discrete minimal surface if and only if there exists a spherical representation $\Sigma_\circ$ such that the corresponding dilation factor $Q(c)$ is stationary on Σ with respect to c. In this case, the dilation factor attains a local maximum on Σ.*

Proof. It is required to show that $\mathcal{K} < 0$ if $\mathcal{M} = 0$, that is

$$H_2K_\circ + K_1H_\circ + H_{\circ 2}K + K_{\circ 1}H = 0. \tag{4.21}$$

To this end, it is first observed that the above condition implies that

$$HH_\circ = \alpha(n_1), \qquad KK_\circ = -\beta(n_2) \tag{4.22}$$

with $\alpha, \beta \neq 0$ for non-degenerate spherical representations. These first integrals are particular cases of those obtained in the context of the integrable class of discrete O surfaces discussed in [17]. However, their validity may be shown directly on use of the linear systems (4.5) and (4.11). Elimination of $H_\circ$ and $K_\circ$ in (4.21) via (4.22) then leads to the relation

$$(\beta H_2H - \alpha K_1K)(H_2K_1 + HK) = 0 \tag{4.23}$$

so that

$$\frac{H_2H}{K_1K} = \frac{\alpha}{\beta} > 0 \tag{4.24}$$

by virtue of the non-degeneracy and embeddedness assumptions. Finally, elimination of $H_\circ$ and $K_\circ$ from the expression (4.13) for the discrete Gaußian curvature indeed reveals that

$$\mathcal{K} = -\frac{\alpha\beta}{H_2 H K_1 K} < 0. \qquad \square \tag{4.25}$$

5. Connections: the 'parallel surface method'

The representation (2.9) demonstrates that the mean curvature $\mathcal{M}$ of a surface Σ may be obtained from Σ and two parallel surfaces $\Sigma^{\parallel}(\pm c)$ without (explicit) reference to the Gauß map. Similarly, the Gaußian curvature $\mathcal{K}$ admits the 'second-order central difference' formulation

$$\mathcal{K} = \frac{\mathrm{d}A^{\parallel}(c) - 2\,\mathrm{d}A + \mathrm{d}A^{\parallel}(-c)}{2c^2\,\mathrm{d}A}, \tag{5.1}$$

which is also valid in the discrete setting if the symbol 'd' is formally replaced by 'δ'. In general, the dilation factor $Q(c)$ as given by (2.7) allows one to reconstruct algebraically the Gaußian and mean curvatures if the infinitesimal area elements of the surface Σ and two (or more) parallel surfaces are known. This fact lies at the heart of the *parallel surface method* which has been used to determine the (average) interfacial curvatures of 'bicontinuous structures' present in a variety of condensed materials (see [13] and references therein). Measuring the curvatures of the interface formed, for instance, in binary mixtures of polymers is essential since these determine the rate of the domain growth in the formation process.

The parallel surface method makes use of the surface areas A and $A^{\parallel}$ of the interface Σ and a sample of parallel surfaces $\Sigma^{\parallel}$ respectively to obtain algebraically the surface averages

$$\langle\mathcal{K}\rangle = \frac{1}{A}\int \mathcal{K}\,\mathrm{d}A, \qquad \langle\mathcal{M}\rangle = \frac{1}{A}\int \mathcal{M}\,\mathrm{d}A, \quad A = \int \mathrm{d}A \tag{5.2}$$

of the Gaußian and mean curvatures via the dilation factor. In fact, the 'integrated version' of the relationship (2.7) reads

$$\frac{A^{\parallel}}{A} = 1 - 2c\langle\mathcal{M}\rangle + c^2\langle\mathcal{K}\rangle, \tag{5.3}$$

which shows that an extremum principle may also be established for surfaces of vanishing average mean curvature ($\langle\mathcal{M}\rangle = 0$) with respect to the *total* surface area $A^{\parallel}(c)$. Thus, the average mean curvature of a surface vanishes if and only if the apex of the parabola defined by (5.3) is located at $c = 0$ provided that $\langle\mathcal{K}\rangle \neq 0$. In [11], the existence of this parabolic profile has been exploited in the context of the 'spinodal decomposition' into two phases of a mixture consisting of polybutadiene (PB) and poly(styrene-ran-butadiene) (SBR). Remarkably, by means of the parallel surface method, it has been concluded that the average mean curvature of the spinodal interface is approximately zero with the apex of the parabola indeed corresponding to a parallel surface which is very close to the spinodal interface.

In practice, due to the 'digital' nature of the 3D imaging processes involving, for instance, laser scanning confocal microscopy [10] in the case of the condensed matter system discussed above, the interface may not be reconstructed precisely but may only be approximated by a discrete surface. In the current context, the *marching cubes algorithm* [12] is employed to obtain a triangulated interface and corresponding 'normals' which are defined on the vertices of the discrete interface. As in the present paper, a parallel discrete surface is then generated by translating the vertices along the normals by a fixed distance c and the calculation of the average Gaußian and mean curvatures is based on the areas of parallel discrete surfaces. Thus, it is evident that the practical determination of (average) interfacial curvatures is essentially a 'discrete' problem with associated key concepts of 'discrete normals' and 'discrete parallel surfaces.' In this connection, it is observed that the relation (5.3) is also valid in the current discrete setting, where the average Gaußian and mean curvatures are canonically defined by

$$\langle\mathcal{K}\rangle = \frac{1}{A}\sum \mathcal{K}\delta A, \qquad \langle\mathcal{M}\rangle = \frac{1}{A}\sum \mathcal{M}\delta A, \quad A = \sum \delta A. \tag{5.4}$$

References

[1] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[2] A. Bobenko, U. Pinkall, Discrete surfaces with constant negative Gaussian curvature and the Hirota equation, J. Diff. Geom. 43 (1996) 527–611.
[3] A. Bobenko, U. Pinkall, Discrete isothermic surfaces, J. Reine Angew. Math. 475 (1996) 187–208.
[4] A.I. Bobenko, Discrete conformal maps and surfaces, in: P.A. Clarkson, F.W. Nijhoff (Eds.), Symmetries and Integrability of Difference Equations, London Math. Soc. Lecture Note Series 255, Cambridge University Press, 1999, pp. 97–108.
[5] A.I. Bobenko, U. Pinkall, Discretization of surfaces and integrable systems, in: A.I. Bobenko, R. Seiler (Eds.), Discrete Integrable Geometry and Physics, Clarendon Press, Oxford, 1999, pp. 1–58.
[6] A.I. Bobenko, R. Seiler (Eds.), Discrete Integrable Geometry and Physics, Clarendon Press, Oxford, 1999.
[7] M.P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, NJ, 1976.
[8] L.P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Dover Publications, New York, 1960.
[9] S.T. Hyde, Microstructure of bicontinuous surfactant aggregates, J. Phys. Chem. 93 (1989) 1458–1464.
[10] H. Jinnai, Y. Nishikawa, T. Koga, T. Hashimoto, Direct observation of three-dimensional bicontinuous structure developed via spinodal decomposition, Macromolecules 28 (1995) 4782–4784.
[11] H. Jinnai, T. Koga, Y. Nishikawa, T. Hashimoto, S.T. Hyde, Curvature determination of spinodal interface in a condensed matter system, Phys. Rev. Lett. 78 (1997) 2248–2251.
[12] W.E. Lorensen, H.E. Cline, Marching cubes: a high resolution 3D surface construction algorithm, in: Proceedings of the SIGGRAPH'87, Comput. Graphics 21 (1987) 163–169.
[13] Y. Nishikawa, H. Jinnai, T. Koga, T. Hashimoto, S.T. Hyde, Measurements of interfacial curvatures of bicontinuous structure from three-dimensional digital images. 1. A parallel surface method, Langmuir 14 (1998) 1242–1249.
[14] C. Rogers, W.K. Schief, Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002.
[15] R. Sauer, Parallelogrammgitter als Modelle pseudosphärischer Flächen, Math. Z. 52 (1950) 611–622.
[16] R. Sauer, Differenzengeometrie, Springer-Verlag, Berlin, 1970.
[17] W.K. Schief, On the unification of classical and novel integrable surfaces. II. Difference geometry, Proc. Roy. Soc. London A 459 (2003) 373–391.
[18] W. Wunderlich, Zur Differenzengeometrie der Flächen konstanter negativer Krümmung, Österreich, Akad. Math. -Nat. Kl. S.-B. II 160 (1951) 39–77.